4/5/17 $\boxed{\text{I}}$

Def: matrix orthogonal iff $Q^TQ = I$.

ie. cols form orthonormal basis w/ respect to dot product

proper if $\det Q = +1$

- 5.3.1
  - (a)
  - (b)
- 5.3.8
  - (a)
  - (b)
- 5.3.9

5.3.12

- HW 5.3.16
  - (a)
  - (b)

5.3.18

- 5.3.27 (b)
  - HW - (a)
- 5.3.28 (i)
- 5.5.2. (c)

(HW 5.5.3)

- 5.5.4

5.5.7

5.3.1 Determine if
(i) orthogonal
(ii) proper orthogonal

(a) $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \neq I$$

not orthogonal.

(b) $A = \begin{pmatrix} 12/13 & 5/13 \\ -5/13 & 12/13 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 12/13 & -5/13 \\ 5/13 & 12/13 \end{pmatrix}\begin{pmatrix} 12/13 & 5/13 \\ -5/13 & 12/13 \end{pmatrix} = \begin{pmatrix} \frac{169}{169} & 0 \\ 0 & \frac{169}{169} \end{pmatrix} = I$$

orthogonal ✓

$$\det A = \frac{144 + 25}{169} = +1$$ proper orthogonal ✓

5.3.8 (a) prove that the transpose of an orthogonal matrix is also orthogonal.

Proof:
Let A be an orthogonal matrix.
Then $A^T A = I \Leftrightarrow A^T = A^{-1}$ and $(A^T)^{-1} = A$

$$(A^T)^T = A = A^{T^{-1}} \Rightarrow (A^T)^T A^T = I$$

So A^T is orthogonal.

5.3.8 (b) Explain why the rows of an $n \times n$ orthogonal matrix, Q, also form an orthonormal basis of $\mathbb{R}^n$.

Rows of Q are cols of Q^T, which is also orth, by part a $\Rightarrow$ rows form an orthonormal basis of $\mathbb{R}^n$.

5.3.9 Prove that the inverse of an orthogonal matrix is also orthogonal.

Proof:

Let A be an orthogonal matrix.

$$A^T A = I, \quad {}^{*}A^T = A^{-1}, \quad (A^T)^{-1} = A$$

$$(A^{-1})^T = (A^T)^T = A = (A^{-1})^{-1}$$

$$(A^{-1})^T = (A^{-1})^{-1} \Rightarrow (A^{-1})^T A^{-1} = I \quad \checkmark$$

5.3.16. (a) Prove that if Q is an orthogonal matrix, then $\|Qx\| = \|x\|$ for any $x \in \mathbb{R}^n$, where $\| \ \|$ is std Euclidean norm (dot prod).

Assume Q is an orthogonal matrix.

Then $Q^T Q = I$.

$$\|Qx\|^2 = \langle Qx, Qx \rangle \overset{= Qx \cdot Qx}{=} (Qx)^T(Qx) = x^T Q^T Q x = x^T I x = x^T x$$

$$= x \cdot x = \langle x, x \rangle = \|x\|^2$$

$\Rightarrow \|Qx\| = \|x\|$ (since both are pos.)

(5.3.16 (b) Prove that if $\|Qx\| = \|x\|$ for all $x \in \mathbb{R}^n$, then Q is orthogonal.)

$$\|Qx\| = \|x\|$$

$$\Rightarrow \|Qx\|^2 = \|x\|^2$$

$$\Rightarrow \langle Qx, Qx\rangle = \langle x, x\rangle$$

$$\Rightarrow (Qx)\cdot(Qx) = x \cdot x$$

$$\Rightarrow (Qx)^T Qx = x^T x.$$

$$\Rightarrow x^T Q^T Q x = x^T I x$$

use exercise 3.4.19

$$\Rightarrow Q^T Q = I.$$

5.3.27 Find QR factorization

(b) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$

set $r_{11} = \left\| \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right\| = \sqrt{16+9} = \sqrt{25} = 5$

$\div$ 1st col by r_{11}

$$\Rightarrow A = \begin{pmatrix} 4/5 & 3 \\ 3/5 & 2 \end{pmatrix}$$

take dot prod w/ other cols.

$$\begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 12/5 + 6/5 = 18/5 \Rightarrow r_{12} = 18/5$$

2nd col = 2nd col $-$ r_{12} 1st col

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} - 18/5 \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 72/25 \\ 54/25 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2\ 22/25 \\ 2\ 4/25 \end{pmatrix}$$

$$= \begin{pmatrix} 3/25 \\ -4/25 \end{pmatrix}$$

$$A = \begin{pmatrix} 4/5 & 3/25 \\ 3/5 & -4/25 \end{pmatrix}$$

set $r_{22} = \left\| \begin{pmatrix} 3/25 \\ -4/25 \end{pmatrix} \right\| = \sqrt{\frac{9}{125} + \frac{16}{125}} = \sqrt{\frac{1}{25}} = \frac{1}{5}$

normalize 2nd col by $\div$ by r_{22} :

$$Q = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix}$$

$$R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} = \begin{pmatrix} 5 & 18/5 \\ 0 & 1/5 \end{pmatrix}.$$

Check $QR = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix} \begin{pmatrix} 5 & 18/5 \\ 0 & 1/5 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 72/25 + \frac{3}{25} \\ 3 & 54/25 - \frac{4}{25} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} = A \checkmark .$$

(a) HW

5.3.28

(i)

(a) Find QR factorization

(b) use to solve system

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

set $r_{11} = \left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\| = \sqrt{2}$

$\div$ 1st col by r_{11} (normalize)

$$A = \begin{pmatrix} 1/\sqrt{2} & 2 \\ -1/\sqrt{2} & 3 \end{pmatrix}$$

take dot product:

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = r_{12}$$

2nd col $\to$ 2nd col $- r_{12}$ 1st col.

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} - -\frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1/\sqrt{2} & 5/2 \\ -1/\sqrt{2} & 5/2 \end{pmatrix}$$

set $r_{22} = \left\| \begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix} \right\| = \sqrt{\frac{25}{4} + \frac{25}{4}} = \sqrt{50/4} = \frac{5\sqrt{2}}{2} = \frac{5}{\sqrt{2}}$

$\div$ 2nd col by r_{22}

$$Q = \begin{pmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \qquad R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -1/\sqrt{2} \\ 0 & \frac{5}{\sqrt{2}} \end{pmatrix}$$

(b) Solve $Ax = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\Rightarrow x = A^{-1}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= (QR)^{-1}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= R^{-1}Q^{-1}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= R^{-1}Q^{T}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow Rx = Q^{T}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$Q^T\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Solve $Rx = \begin{pmatrix} -3/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$\begin{pmatrix} \sqrt{2} & -1/\sqrt{2} \\ 0 & 5/\sqrt{2} \end{pmatrix} x = \begin{pmatrix} -3/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow 5/\sqrt{2}\, x_2 = 1/\sqrt{2}$$

$$\Rightarrow \boxed{x_2 = 1/5}$$

$$\sqrt{2}x_1 - \frac{1}{\sqrt{2}}\left(\frac{1}{5}\right) = -3/\sqrt{2}$$

$$\Rightarrow \sqrt{2}x_1 = \frac{-15}{5\sqrt{2}} + \frac{1}{5\sqrt{2}}$$

$$\Rightarrow x_1 = \frac{-14}{10}$$

$$\boxed{x_1 = \frac{-7}{5}}$$

$$\boxed{x = \begin{pmatrix} -7/5 \\ 1/5 \end{pmatrix}}$$

5.5.2 (c) Find the orthogonal projection of $\begin{pmatrix}1\\1\\1\end{pmatrix} = v$ onto the plane spanned by $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\2\\1\end{pmatrix}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad v_1 \qquad v_2$

$\left\| \begin{pmatrix}1\\1\\0\end{pmatrix} \right\| = \sqrt{2}$

First check orthogonal:

$\left\| \begin{pmatrix}-2\\2\\1\end{pmatrix} \right\| = \sqrt{9} = 3$

$\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}-2\\2\\1\end{pmatrix} = 0 \checkmark.$

orth. proj. of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ onto $\text{span}\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\2\\1\end{pmatrix} \right\}$ is

$$W = \frac{\langle v, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle v, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= \frac{\left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle}{2} \begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{\left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\1\end{pmatrix} \right\rangle}{9} \begin{pmatrix}-2\\2\\1\end{pmatrix}$$

$$= \frac{2}{2} \begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{9} \begin{pmatrix}-2\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}7/9\\11/9\\1/9\end{pmatrix}.$$

5.5.4. Find orth. proj. of $v = \begin{pmatrix}1\\3\\-1\end{pmatrix}$ onto $\text{span}\left\{ \begin{pmatrix}-1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-3\end{pmatrix} \right\}$ by first using Gram-Schmidt to find an orthogonal basis:

$v_1 = \begin{pmatrix}-1\\2\\1\end{pmatrix}$

$$v_2 = \begin{pmatrix}2\\1\\-3\end{pmatrix} - \frac{\left\langle \begin{pmatrix}-1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-3\end{pmatrix} \right\rangle}{\left\| \begin{pmatrix}-1\\2\\1\end{pmatrix} \right\|^2} \begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-3\end{pmatrix} - \frac{-3}{6} \begin{pmatrix}-1\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}3/2\\2\\-5/2\end{pmatrix}$$

Orth. basis : $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3/2 \\ 2 \\ -5/2 \end{pmatrix} \right\}$

$\qquad\qquad v_1 \qquad\qquad v_2$

$$\|v_1\| = \sqrt{6}$$

$$\|v_2\| = \sqrt{9/4 + 4 + 25/4} = \sqrt{\frac{9+16+25}{4}} = \sqrt{\frac{50}{4}} = \frac{\sqrt{50}}{2}$$

orth proj. is

$$w = \frac{\langle v, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle v, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= \frac{\left\langle \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\rangle}{6} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \frac{\left\langle \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 3/2 \\ 2 \\ -5/2 \end{pmatrix} \right\rangle}{50/4} \begin{pmatrix} 3/2 \\ 2 \\ -5/2 \end{pmatrix}$$

$$= \frac{4}{6} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \frac{3/2 + 6 + 5/2}{50/4} \begin{pmatrix} 3/2 \\ 2 \\ -5/2 \end{pmatrix}$$

$$= 2/3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \frac{20/2}{50/4} \begin{pmatrix} 3/2 \\ 2 \\ -5/2 \end{pmatrix}$$

$$= 2/3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \frac{4}{5} \begin{pmatrix} 3/2 \\ 2 \\ -5/2 \end{pmatrix}$$